

Gradient metasurfaces as slowly varying diffraction gratings

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Phase-gradient metasurfaces have emerged in the past decade as a powerful tool to manipulate light on flat surfaces. These surfaces introduce new “*generalized laws of reflection and refraction*,” and right from their beginning there has been an interesting discussion as to the degree to which they resemble, or not, reflect-/transmit-arrays or diffraction gratings. On the basis of an appropriate, fully analytic but physically intuitive diffraction model, we here show rigorously that the metasurfaces’ reflection/refraction laws are mathematically akin to *phase laws* on the surfaces of traditional diffraction gratings. The developed theory, in fact, allows for even more general laws of reflection and refraction, encompassing both propagating and evanescent waves, as well as for calculating the diffracted field (both reflected and refracted) to all orders. We also point out and analytically quantify the precise limit under which a slowly varying diffraction grating becomes a phase-gradient metasurface. Our closed-form analysis reveals that the “*generalized laws of reflection and refraction*” of metasurfaces can complementarily be understood and derived in the limit of the established theory of surface-relief diffraction gratings (SRGs). © 2026 Optica Publishing Group under the terms of the [Optica Open Access Publishing Agreement](#)

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1. INTRODUCTION

Engineered surfaces with sub-wavelength resonant elements placed on them, known as metasurfaces, have in the past 10–15 years allowed for a cornucopia of useful applications, heralding the dynamic field of “flat optics.” Their particularly appealing feature is that they allow for almost complete control of light (or other waves) simply by modifying a flat surface, i.e., without requiring a bulk, three-dimensional (meta)material. In particular, they have been shown to be capable of judiciously manipulating the amplitude, phase, frequency, polarization, chirality, temporal or nonlinear characteristics, and directions of reflection and refraction of a lightwave—in both the classical and quantum regimes. Especially enticing are the so-called phase-gradient metasurfaces—historically, the first type of widely used metasurfaces—where the sizes and shapes of the resonant elements are changing gradually within one wavelength of the incident radiation, giving rise, among others, to completely flat—rather than curved—lenses or to optical vortices.

A conventional surface-relief diffraction grating (SRG) and a phase-gradient metasurface differ primarily in how their optical response is implemented. In a relief grating, the position of a material boundary is modulated, usually periodically, so the accumulated phase, Fresnel response, and order amplitudes are

tied to the height profile and constituent materials. A phase-gradient metasurface is nominally planar and patterned with subwavelength scatterers whose geometry can prescribe a local phase and, in more general implementations, amplitude and polarization response. These two categories indeed overlap: a periodically repeated metasurface supercell is also a diffraction grating at the level of its global Floquet spectrum. Nevertheless, the conventional relief grating is a transparent reference system that separates common tangential-momentum kinematics from the different electromagnetic mechanisms and design freedoms of the two platforms.

Gradient metasurfaces have introduced new, so-called, generalized laws for the reflection and refraction of a lightwave from an interface with smoothly varying refractive index. These laws were derived in Ref. [1] from a simple, physically intuitive application of Fermat’s principle, and in mathematical terms, they are the following:

$$\sin(\vartheta_r) - \sin(\vartheta_i) = \frac{\lambda_0}{2\pi n_i} \frac{d\Phi}{dz}, \quad (1)$$

$$\sin(\vartheta_t)n_t - \sin(\vartheta_i)n_i = \frac{\lambda_0}{2\pi} \frac{d\Phi}{dz}, \quad (2)$$

where ϑ_t , ϑ_i , ϑ_r are, respectively, the angle of transmission, incidence, and reflection, n_t , n_i are, respectively, the refractive index

of the medium in which refraction or incidence takes place, and Φ is the local phase shift along the smoothly varying (across z) interface.

The connection between generalized phase-gradient relations and diffraction theory is not itself new [2–4]. Yu *et al.* introduced the interfacial phase-gradient laws [1,5]; Larouche and Smith established a formal equivalence between generalized refraction and a selected order of a blazed grating [6]; classical grating theory supplies Floquet momenta and their propagating or evanescent Rayleigh orders [7–9]; and reflectarrays and metasurface reflectors have been discussed from a unified grating perspective [10]. Recent work has further combined phase-gradient control with Floquet spatial harmonics [11] and examined evanescent-wave diffraction by phase-gradient metasurfaces [12].

Here, we address in a unified fashion and in some detail the following issue: within an explicitly defined scalar, local-tangent, small-slope approximation for a single-valued relief $x = h(z)$, we derive *in parallel* the distinct height-to-phase maps for reflection and transmission; connect the local phase-gradient and global periodic-order descriptions in one angular-spectrum formulation; and follow the resulting tangential momentum through the propagating–evanescent threshold. The result is a *leading-order kinematic correspondence*, where the analysis (below) is an asymptotically controlled, leading-order construction. We first map a gently varying relief onto equivalent reflection and transmission phase screens, then recover the periodic Floquet orders and the local stationary-phase laws, and finally continue those momenta across the light line. The approximation and its leading neglected terms are stated explicitly below.

Consider a monochromatic field with time dependence $e^{-i\omega t}$. Let $k_0 = \frac{\omega}{c} = \frac{2\pi}{\lambda_0}$, $k_j = n_j k_0$, where n_1 is the refractive index of the incident medium and n_2 the refractive index of the transmitted medium. The interface is not flat; it is a single-valued height profile $x = h(z)$ (see

Fig. 1). We assume throughout a single-valued smooth profile $h(z)$ and that we operate in the small-slope/smooth-grating regime, i.e., $|h'(z)| \ll 1$. The incident field in medium 1 is taken as a plane wave propagating toward $+x$: $E_i(x, z) = E_0 e^{i(k_{1x}x + q_i z)}$, with $k_{1x} = k_1 \cos \theta_i$, $q_i = k_1 \sin \theta_i$, where θ_i is the angle of incidence measured from the mean normal $+x$. For the corresponding perfectly flat interface $x = 0$, we assume that the reflected and transmitted reference waves are $E_r^{(0)}(x, z) = r_0 E_0 e^{i(-k_{1x}x + q_i z)}$, $x < 0$, and $E_t^{(0)}(x, z) = t_0 E_0 e^{i(k_{2x}^{(0)}x + q_i z)}$, $x > 0$, where $k_{2x}^{(0)} = k_2 \cos \theta_t^{(0)}$, and $n_1 \sin \theta_i = n_2 \sin \theta_t^{(0)}$. Here, r_0, t_0 are the usual Fresnel coefficients for the chosen polarization at the flat interface. Their exact form is not needed for the angular derivation that we pursue.

The grating considered here is a surface-relief boundary, i.e., not a film or ribbon of constant thickness. Medium 1 occupies $x < h(z)$, medium 2 occupies $x > h(z)$, and $x = h(z)$ is their sole material interface. The function $h(z)$ is the (signed) displacement of that boundary along the mean-surface normal x from the reference plane $x = 0$. For the sinusoidal example, $h(z) = h_0 \sin(Kz)$, $K = 2\pi/\Lambda$; h_0 is the peak relief from the mean plane and $2h_0$ is the peak-to-valley height. The fixed r_0 and t_0 used below are reference flat-interface coefficients; their slope-dependent corrections are quantified explicitly.

In each homogeneous half-space, the field satisfies the Helmholtz equation. Therefore, in two dimensions, any reflected or transmitted field can be written as an angular-spectrum superposition of plane waves:

$$E_r(x, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}_r(q) e^{-ik_1(q)x} e^{iqz} dq, \quad x < 0, \quad (3)$$

$$E_t(x, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}_t(q) e^{+ik_2(q)x} e^{iqz} dq, \quad x > 0, \quad (4)$$

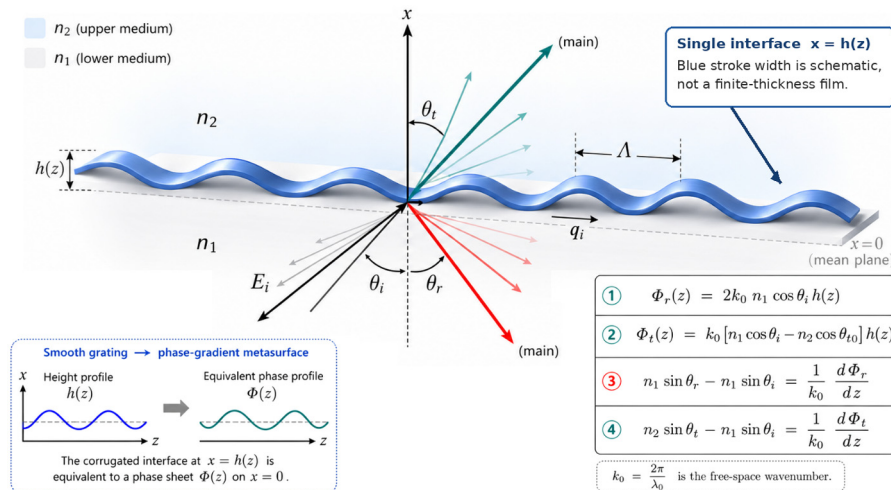


Fig. 1. Geometric representation of a single corrugated material interface. Medium 1 occupies $x < h(z)$ and medium 2 occupies $x > h(z)$; the blue curve denotes the boundary $x = h(z)$, i.e., not a separate finite-thickness layer. The quantity $h(z)$ is the (signed) displacement from $x = 0$. For the sinusoid shown, $h(z) = h_0 \sin(2\pi z/\Lambda)$, where h_0 is the peak amplitude and $2h_0$ is the peak-to-valley relief. Incident light is decomposed into reflected and transmitted propagating or evanescent orders. The inset shows the approximate mapping onto the equivalent reflection and transmission phase screens defined in the text. The case of nonperiodic variations at the media interfaces is also studied in the main text.

with $\kappa_j(q) = \sqrt{k_j^2 - q^2}$, $\Re \kappa_j \geq 0$, $\Im \kappa_j \geq 0$. Thus, if $|q| < k_j$, then $\kappa_j(q) \in \mathbb{R}$: the component is propagating, whereas if $|q| > k_j$, then $\kappa_j(q) = i\alpha_j(q)$ with $\alpha_j(q) = \sqrt{q^2 - k_j^2} > 0$, and the component is evanescent. As a result, as we shall see, for our purposes here, the whole problem reduces to finding the tangential spectrum of $\tilde{E}_r(q)$, $\tilde{E}_t(q)$.

The key step in our analysis now is to rewrite the (spatially varying) height of the grating as an effective phase profile on the mean plane $x = 0$. Then, at the actual boundary point $(x, z) = (h(z), z)$, the incident field becomes $E_i(h(z), z) = E_0 e^{i(q_i z + k_{1x} h(z))}$. In the tangent-plane approximation, the boundary conditions are applied locally as if each point of the interface were a locally flat interface. Therefore, the boundary values of the reflected and transmitted fields are $E_r(h(z), z) = r_0 E_0 e^{i(q_i z + k_{1x} h(z))}$ and $E_t(h(z), z) = t_0 E_0 e^{i(q_i z + k_{1x} h(z))}$. The only new ingredient relative to the flat interface is that the boundary has now moved from $x = 0$ to $x = h(z)$. That displacement produces a phase of pure geometric/structural origin, i.e., from the varying $h(z)$.

A reflected wave traveling in medium 1 from the displaced point $x = h(z)$ back to the mean plane $x = 0$ accumulates the phase $+k_{1x} h(z)$. Therefore, $E_r(0, z) = E_r(h(z), z) e^{+ik_{1x} h(z)}$, or, as explained above, $E_r(0, z) = r_0 E_0 e^{i(q_i z + 2k_{1x} h(z))}$. We may thus define the height-induced reflection phase:

$$\Phi_r(z) = 2k_{1x} h(z) = 2k_0 n_1 \cos \theta_i h(z), \quad (5)$$

$$E_r(0, z) = r_0 E_0 e^{i[q_i z + \Phi_r(z)]}. \quad (6)$$

Within the tangent-plane phase-screen approximation, this has the same kinematic form as a metasurface-imposed reflection phase profile. Here, however, the phase is constrained by the physical surface displacement.

Likewise, a transmitted wave traveling in medium 2 from the displaced boundary point $x = h(z)$ back to the mean plane $x = 0$ picks up a phase $-k_{2x}^{(0)} h(z)$. Thus, $E_t(0, z) = E_t(h(z), z) e^{-ik_{2x}^{(0)} h(z)}$, and using the previous expression for $E_t(h(z), z)$, we get: $E_t(0, z) = t_0 E_0 e^{i(q_i z + (k_{1x} - k_{2x}^{(0)})h(z))}$. We may now, as above, define a height-induced transmission phase:

$$\Phi_t(z) = (k_{1x} - k_{2x}^{(0)})h(z) = k_0(n_1 \cos \theta_i - n_2 \cos \theta_t^{(0)})h(z), \quad (7)$$

From where it follows that

$$E_t(0, z) = t_0 E_0 e^{i[q_i z + \Phi_t(z)]}. \quad (8)$$

Within the same approximation, the transmission expression likewise has the phase-screen form used in generalized refraction models, while its amplitude and vector boundary response need not coincide with those of a metasurface.

2. DOMAIN AND ACCURACY OF THE LOCAL PHASE-SCREEN APPROXIMATION

The fields on $x = 0$ are equivalent fields back-projected to a reference phase screen; when a sign-changing relief crosses the mean plane, they are not simultaneous physical fields in one homogeneous medium at every z . Our construction retains the

displacement phase generated by $h(z)$ while evaluating Fresnel coefficients at the reference incidence angle.

Let $s = h'(z)$, $q_i = k_1 \sin \theta_i$, $\kappa_{1i} = k_1 \cos \theta_i$, and $\kappa_{20} = k_2 \cos \theta_t^{(0)}$. Since $\theta_{i,loc} = \theta_i + \arctan(s)$, the local coefficients satisfy $r_{loc} = r_0 + (dr_F/d\theta_i)s + O(s^2)$ and $t_{loc} = t_0 + (dt_F/d\theta_i)s + O(s^2)$. Holding r_0 and t_0 fixed therefore neglects $O(s)$ pointwise amplitude corrections. By contrast, exact local ray geometry gives $q_{r,loc} = q_i + 2\kappa_{1i}s - 2q_i s^2 + O(k_1 s^3)$ and $q_{t,loc} = q_i + (\kappa_{1i} - \kappa_{20})s + q_i(\kappa_{1i}/\kappa_{20} - 1)s^2 + O(k_1 s^3)$. Thus, the phase-gradient momentum laws are correct through first order in slope, with their first angular corrections at $O(s^2)$.

The local-plane approximation also requires $k_j R \gg 1$, where $R = [1 + h'(z)^2]^{3/2}/|h''(z)|$, together with negligible shadowing and repeated scattering. It is nonuniform near grazing incidence, Brewster zeros, critical angles, Rayleigh/Wood anomalies, sharp corners, and resonant or multiply scattering grooves. The full factors $\exp[i\Phi_{r,t}(z)]$ are retained, so no expansion in $k_0 h$ is made and h_0 need not itself be deeply subwavelength. A wavelength-scale height can remain within the model only if it varies sufficiently gently. For $h(z) = h_0 \sin(2\pi z/\Lambda)$, $\max|h'| = 2\pi h_0/\Lambda$ and, at small slope, $R_{min} \approx \Lambda^2/(4\pi^2 h_0)$. Outside these conditions, the exact periodic Floquet positions remain fixed by symmetry, but the geometric phase, local-angle law, and order amplitudes require RCWA, a boundary-integral method, FEM, or FDTD [7–9]. The relief displacement $h(z)$ is also distinct from the physical thickness of a metasurface scatterer. A wavelength-thick or resonant meta-atom may still admit a calibrated local response $A(z)\exp[i\Phi(z)]$ and the same kinematic momentum rule, but Eqs. (5) and (7) do not infer that phase from the meta-atom height.

At this point, for simplicity (we examine the most general case too, later on) and for clearly illustrating the main physical points, let us for the moment assume that the smooth grating varies sinusoidally, i.e., $h(z) = h_0 \sin(Kz)$, with $K = \frac{2\pi}{\Lambda}$. Then, the height-induced phases are also sinusoidal: $\Phi_r(z) = \Delta\Phi_r \sin(Kz)$, with $\Delta\Phi_r = 2k_{1x}h_0 = 2k_0 n_1 h_0 \cos \theta_i$, and $\Phi_t(z) = \Delta\Phi_t \sin(Kz)$, with $\Delta\Phi_t = (k_{1x} - k_{2x}^{(0)})h_0 = k_0(n_1 \cos \theta_i - n_2 \cos \theta_t^{(0)})h_0$. Hence, the reflected and transmitted fields at the mean plane are, respectively: $E_r(0, z) = r_0 E_0 e^{iq_i z} e^{i\Delta\Phi_r \sin(Kz)}$ and $E_t(0, z) = t_0 E_0 e^{iq_i z} e^{i\Delta\Phi_t \sin(Kz)}$. Using the Jacobi–Anger expansion, $e^{ia \sin u} = \sum_{v=-\infty}^{\infty} J_v(a) e^{iv u}$, where J_v is the Bessel function of the first kind, we then immediately get: $E_r(0, z) = r_0 E_0 \sum_{v=-\infty}^{\infty} J_v(\Delta\Phi_r) e^{i(q_i + vK)z}$ and $E_t(0, z) = t_0 E_0 \sum_{v=-\infty}^{\infty} J_v(\Delta\Phi_t) e^{i(q_i + vK)z}$. Substituting these two expressions into the angular-spectrum representation yields: $E_r(x, z) = r_0 E_0 \sum_{v=-\infty}^{\infty} J_v(\Delta\Phi_r) e^{-ik_{1,v} x} e^{iq_v z}$ for $x < 0$, and $E_t(x, z) = t_0 E_0 \sum_{v=-\infty}^{\infty} J_v(\Delta\Phi_t) e^{+ik_{2,v} x} e^{iq_v z}$, for $x > 0$, where $q_v = q_i + vK$, $\kappa_{1,v} = \sqrt{k_1^2 - q_v^2}$, and $\kappa_{2,v} = \sqrt{k_2^2 - q_v^2}$.

3. ORDER AMPLITUDES AND POWER BALANCE

The Jacobi–Anger coefficients $J_v(a_r)$ and $J_v(a_t)$ are equivalent-plane field amplitudes, i.e., not power fractions. For a

TE scalar electric-field convention, the approximate flux efficiencies are $R_v = [\text{Re}(\kappa_{1,v})/\kappa_{1i}][r_0 J_v(a_i)]^2$ and $T_v = [\mu_1 \text{Re}(\kappa_{2,v})/(\mu_2 \kappa_{1i})][t_0 J_v(a_i)]^2$. Other field conventions require the corresponding TE or TM modal admittance factor.

Only propagating orders contribute to outward time-averaged power; evanescent orders may have a nonzero boundary amplitude but carry no net normal power in a lossless homogeneous half-space. Although $\Sigma_v |J_v(a)|^2 = 1$ for real a , this preserves an equivalent-plane squared-amplitude norm over all Fourier components. It does not imply $\Sigma R_v + \Sigma T_v = 1$, because the modal admittances vary with order, and the constant r_0, t_0 screen does not enforce all vector boundary conditions.

For the zero-mean sinusoidal profile, sufficiently far from a light line and with all appreciably populated orders propagating, expansion of $\kappa_j(q_i + \nu K)$ and the Bessel moments $\Sigma \nu |J_\nu|^2 = 0$ and $\Sigma \nu^2 |J_\nu|^2 = a^2/2$ shows that the first-order contribution cancels and the residual power imbalance begins at $O[(Kh_0)^2]$, namely second order in maximum slope. This expansion is nonuniform at the Rayleigh cutoff. Exact conservation and quantitative efficiencies require the complete electromagnetic boundary-value problem.

If $|q_v| \leq k_1$, then the ν -th *reflected* order is propagating, and $q_v = k_1 \sin \theta_{r,v}$. Using $q_i = k_1 \sin \theta_i$, we then get

$$\sin \theta_{r,v} = \sin \theta_i + \frac{\nu K}{k_1} = \sin \theta_i + \nu \frac{\lambda_0}{n_1 \Lambda}. \quad (9)$$

Likewise, if $|q_v| \leq k_2$, then the ν -th *transmitted* order is propagating, and $q_v = k_2 \sin \theta_{t,v}$. Hence,

$$\sin \theta_{t,v} = \frac{q_i + \nu K}{k_2} = \frac{n_1}{n_2} \sin \theta_i + \nu \frac{\lambda_0}{n_2 \Lambda}. \quad (10)$$

Note the striking resemblance of these reflection and refraction grating laws to the generalized laws of reflection and refraction describing phase-gradient metasurfaces in Eqs. (1) and (2).

If $|q_v| > k_1$, then the *reflected* order is evanescent: $\kappa_{1,v} = i\alpha_{1,v}$, $\alpha_{1,v} = \sqrt{q_v^2 - k_1^2}$, and its field is $E_{r,v}(x, z) \propto e^{\alpha_{1,v}x} e^{iq_v z}$, for $x < 0$, which indeed decays as $x \rightarrow -\infty$. Similarly, if $|q_v| > k_2$, then the *transmitted* order is evanescent: $\kappa_{2,v} = i\alpha_{2,v}$, $\alpha_{2,v} = \sqrt{q_v^2 - k_2^2}$, and $E_{t,v}(x, z) \propto e^{-\alpha_{2,v}x} e^{iq_v z}$, for $x > 0$, which decays as $x \rightarrow +\infty$.

To see even more clearly the conspicuous similarity between Eqs. (9), (10) and, correspondingly, Eqs. (1) and (2), we may differentiate the total phase at the mean plane. For *reflection*, we have: $\psi_r(z) = q_i z + \Phi_r(z) = q_i z + \Delta \Phi_r \sin(Kz)$, so the local tangential wavevector is $q_{r,\text{loc}}(z) = \psi'_r(z) = q_i + \Delta \Phi_r K \cos(Kz)$. Since $\Delta \Phi_r = 2k_0 n_1 h_0 \cos \theta_i$, we get: $q_{r,\text{loc}}(z) = k_1 \sin \theta_i + 2k_1 h_0 K \cos \theta_i \cos(Kz)$. Thus,

$$n_1 \sin \theta_r(z) - n_1 \sin \theta_i = \frac{1}{k_0} \frac{d\Phi_r}{dz} = 2n_1 \cos \theta_i h'(z). \quad (11)$$

For *transmission*, we likewise have: $\psi_t(z) = q_i z + \Phi_t(z) = q_i z + \Delta \Phi_t \sin(Kz)$, hence $q_{t,\text{loc}}(z) = \psi'_t(z) = q_i + \Delta \Phi_t K \cos(Kz)$, or equivalently

$$n_2 \sin \theta_t(z) - n_1 \sin \theta_i = \frac{1}{k_0} \frac{d\Phi_t}{dz} = (n_1 \cos \theta_i - n_2 \cos \theta_i^{(0)}) h'(z). \quad (12)$$

Equations (11) and (12)—here derived for familiar, smooth gratings—are of the same leading-order kinematic form as Eqs. (1) and (2) governing phase-gradient metasurfaces. The important point to notice is that locally, a smooth sinusoidal grating behaves indeed like a phase-gradient interface, whereas globally, because (in this present example) the phase gradient is periodic, the far field is quantized into the discrete diffraction orders $q_i + \nu K$. This is the precise, quantifiable, relation for the transition from a continuous phase-gradient law (metasurfaces) to the discrete νK law (diffraction gratings)—showing clearly that conventional, smooth, gratings too are governed by the generalized laws of reflection and refraction regulating metasurfaces.

4. LOCAL PHASE GRADIENT AND GLOBAL DIFFRACTION ORDERS

The quantity $q_{\text{loc}}(z) = q_i + \Phi'(z)$ is the local spatial frequency of the equivalent phase screen. For a periodic profile, however, $\exp[i\Phi(z)]$ is periodic and has a Fourier series $\Sigma_v c_v \exp(i\nu Kz)$, so the global spectrum of an infinite grating is restricted to $q_v = q_i + \nu K$. These descriptions are complementary: $c_v = (1/\Lambda) \int \exp\{i[\Phi(z) - \nu Kz]\} dz$ has stationary points satisfying $\Phi'(z_s) = \nu K$. The local gradient identifies the surface regions contributing by the stationary phase to a permitted order, while coherent repetition enforces the reciprocal-lattice momenta. Thus, q_{loc} is not a continuum of independent far-field channels. A finite array of N periods broadens each peak on a scale of order $2\pi/(N\Lambda)$; a finite aperiodic profile can instead possess a genuinely continuous angular spectrum.

Let us now examine the most general case, where $h(z)$ is an arbitrary (i.e., not necessarily sinusoidal or periodic as before) smooth function. The mean-plane fields remain $E_r(0, z) = r_0 E_0 e^{i[q_i z + \Phi_r(z)]}$, with $\Phi_r(z) = 2k_0 n_1 \cos \theta_i h(z)$, and $E_t(0, z) = t_0 E_0 e^{i[q_i z + \Phi_t(z)]}$, with $\Phi_t(z) = k_0(n_1 \cos \theta_i - n_2 \cos \theta_i^{(0)})h(z)$. Now suppose that $h(z)$ is considered within the slowly varying surface-relief regime stated above. The reflected angular spectrum is $\tilde{E}_r(q) = r_0 E_0 \int_{-\infty}^{\infty} e^{i[\Phi_r(z) + (q_i - q)z]} dz$, while the transmitted angular spectrum is $\tilde{E}_t(q) = t_0 E_0 \int_{-\infty}^{\infty} e^{i[\Phi_t(z) + (q_i - q)z]} dz$. Let $S_r(z; q) = \Phi_r(z) + (q_i - q)z$. By demanding (for these $\tilde{E}_r(q)$, $\tilde{E}_t(q)$ oscillatory integrals, in the smooth-profile limit) that $\frac{dS_r}{dz}(z_s; q) = 0$, we get $q = q_i + \Phi'_r(z_s)$, and since $\Phi'_r(z) = 2k_0 n_1 \cos \theta_i h'(z)$ we find that $q = k_1 \sin \theta_i + 2k_1 \cos \theta_i h'(z_s)$. Identifying $q = k_1 \sin \theta_r$, this then directly gives

$$n_1 \sin \theta_r(z_s) - n_1 \sin \theta_i = \frac{1}{k_0} \frac{d\Phi_r}{dz}(z_s) = 2n_1 \cos \theta_i h'(z_s), \quad (13)$$

which is precisely the smooth-grating analog of the generalized law of *reflection* for *propagating* waves.

Further, for the transmitted field, we assume that $S_t(z; q) = \Phi_t(z) + (q_i - q)z$, and by again demanding that $\frac{dS_t}{dz}(z_s; q) = 0$, we get $q = q_i + \Phi'_t(z_s)$. Since $\Phi'_t(z) = k_0(n_1 \cos \theta_i - n_2 \cos \theta_i^{(0)})h'(z)$, it is $q = k_1 \sin \theta_i +$

$k_0(n_1 \cos \theta_i - n_2 \cos \theta_t^{(0)})h'(z_s)$, and with $q = k_2 \sin \theta_t$, we finally obtain

$$n_2 \sin \theta_t(z_s) - n_1 \sin \theta_i = \frac{1}{k_0} \frac{d\Phi_t}{dz}(z_s) = (n_1 \cos \theta_i - n_2 \cos \theta_t^{(0)})h'(z_s), \quad (14)$$

which is the smooth-grating analog of the generalized law of refraction of metasurfaces for propagating waves.

5. PIECEWISE-SMOOTH AND FABRICATED PROFILES

A measured, faceted, or staircased profile $h_f(z)$ can still be substituted directly into the equivalent mean-plane fields and Fourier transformed as $\exp[i\beta_p h_f(z)]$, with $\beta_r = 2k_{1x}$ and $\beta_t = k_{1x} - k_{2x}^{(0)}$. The differential laws apply to each smooth segment. At a step or corner, however, $h'(z)$ and a unique local incidence angle do not exist; the distributional derivative must not be interpreted as an infinite physical deflection angle. The phase jump adds high spatial frequencies to the screen spectrum, while the actual sidewall or corner also produces vectorial edge diffraction and multiple scattering omitted here.

If $h_f(z) = h(z) + \delta h(z)$, the phase-screen errors are $\delta\Phi_r = 2k_{1x}\delta h$ and $\delta\Phi_t = (k_{1x} - k_{2x}^{(0)})\delta h$. A staircase therefore approaches the intended low-order spectrum when its lateral sampling resolves the target phase, and these phase errors remain small. Coarse height quantization, abrupt sidewalls, or wavelength-scale rounding can redistribute power into parasitic orders; quantitative efficiencies then require a rigorous grating or full-wave method [7–9].

We may now further generalize these laws for evanescent waves too (not studied in Ref. [1]). Indeed, let us keep the same, fixed tangential momentum $q_r(z_s) = q_i + \Phi'_r(z_s) = k_1 \sin \theta_i + 2k_1 \cos \theta_i h'(z_s)$, but now assume that it lies outside the light cone of medium 1, i.e., $|q_r(z_s)| > k_1$. Then, the reflected wave is evanescent, or, equivalently: $|\sin \theta_i + 2h'(z_s) \cos \theta_i| > 1$. We may thus write $\kappa_1(q_r(z_s)) = i\alpha_r(z_s)$ with $\alpha_r(z_s) = \sqrt{q_r(z_s)^2 - k_1^2} = n_1 k_0 \sqrt{(\sin \theta_i + 2h'(z_s) \cos \theta_i)^2 - 1}$. Therefore, the local evanescent reflected field is $E_r(x, z) \sim A_r(z_s) e^{\alpha_r(z_s)x} e^{iq_r(z_s)z}$, $x < 0$, so it decays away from the interface. As a result, the evanescent reflection law for a smooth grating is simply the same generalized reflection law [Eq. (13)]:

$$n_1 \sin \theta_r(z_s) - n_1 \sin \theta_i = \frac{1}{k_0} \Phi'_r(z_s), \theta_r(z_s) \in \mathbb{C}, \quad (15)$$

with the additional, simultaneous condition (evanescence condition) that $|\sin \theta_i + 2h'(z_s) \cos \theta_i| > 1$. Here, if desired, we can also define $\theta_r(z_s) = \text{sgn}(q_r) \frac{\pi}{2} - i \text{sgn}(q_r) \chi_r(z_s)$, so that $\cosh \chi_r(z_s) = |\sin \theta_i + 2h'(z_s) \cos \theta_i|$ and $\alpha_r(z_s) = n_1 k_0 \sinh \chi_r(z_s)$.

Similarly, for the transmitted evanescent field, we have $q_t(z_s) = q_i + \Phi'_t(z_s) = k_1 \sin \theta_i + (k_1 \cos \theta_i - k_2 \cos \theta_t^{(0)})h'(z_s)$. The transmitted field becomes evanescent when $|q_t(z_s)| > k_2$, that is $|\frac{n_1}{n_2} \sin \theta_i + \frac{n_1 \cos \theta_i - n_2 \cos \theta_t^{(0)}}{n_2} h'(z_s)| > 1$. Then, $\kappa_2(q_t(z_s)) = i\alpha_t(z_s)$, with $\alpha_t(z_s) = \sqrt{q_t(z_s)^2 - k_2^2} =$

$n_2 k_0 \sqrt{\left[\frac{n_1}{n_2} \sin \theta_i + \frac{n_1 \cos \theta_i - n_2 \cos \theta_t^{(0)}}{n_2} h'(z_s) \right]^2 - 1}$. Therefore, the local transmitted field is now $E_t(x, z) \sim A_t(z_s) e^{-\alpha_t(z_s)x} e^{iq_t(z_s)z}$, $x > 0$, as a result of which the evanescent refraction law for a smooth grating is

$$n_2 \sin \theta_t(z_s) - n_1 \sin \theta_i = \frac{1}{k_0} \Phi'_t(z_s), \theta_t(z_s) \in \mathbb{C}, \quad (16)$$

with the additional, simultaneous condition (evanescence condition) that $|\frac{n_1}{n_2} \sin \theta_i + \frac{n_1 \cos \theta_i - n_2 \cos \theta_t^{(0)}}{n_2} h'(z_s)| > 1$. Here, too, we can also write $\theta_t(z_s) = \text{sgn}(q_t) \frac{\pi}{2} - i \text{sgn}(q_t) \chi_t(z_s)$, so that $\cosh \chi_t(z_s) = |\frac{n_1}{n_2} \sin \theta_i + \frac{n_1 \cos \theta_i - n_2 \cos \theta_t^{(0)}}{n_2} h'(z_s)|$, where $\alpha_t(z_s) = n_2 k_0 \sinh \chi_t(z_s)$.

For either output medium the signed, decay-compatible branch is equivalently $\theta_j = \text{sgn}(q_j)[\pi/2 - i\chi_j]$. The additional sign multiplying χ_j is essential when $q_j < 0$ because it selects the normal imaginary wave-vector component that decays away from the interface.

To illustrate the reduced-model correspondence between phase-gradient metasurfaces and slow diffraction gratings, Fig. 2 compares the generalized metasurface reflection law with the reflected angular spectrum calculated for a finite smooth grating over the frequency interval 180–360 THz. For the $\nu = +1$ order, Eqs. (1) and (9) give the tangential-momentum relation $q_{r,+1}/k_1 = \sin \theta_i + \lambda_0/(n_1 \Lambda)$, represented by the solid curve. The symbols were obtained by numerically Fourier transforming the finite-aperture mean-plane phase-screen field $E_r(0, z) = r_0 E_0 \exp[iq_i z + i\Delta\Phi_r \sin(Kz)]$ and locating the corresponding angular-spectrum maximum. Their deviation from the analytic prediction remains below 8×10^{-4} , consistent with the finite spectral resolution. Importantly, plotting $q_{r,+1}/k_1$, rather than only a real angle, permits the law to be followed continuously through the light line: below $f_c = 224.7$ THz, it is $q_{r,+1}/k_1 > 1$ and the order is evanescent, whereas above this cutoff a real reflected angle emerges and decreases smoothly with frequency. Thus, the smooth grating and the metasurface formulations remain kinematically equivalent on both sides of the propagating–evanescent transition.

The corresponding comparison for refraction is shown in Fig. 3, where both positive and negative transmitted diffraction orders are considered. For $n_1 = 1$, $n_2 = 1.5$, $\theta_i = 20^\circ$, and $\Lambda = 1.15 \mu\text{m}$, the generalized metasurface law and the smooth-grating law reduce to the common expression $q_{t,\nu}/k_2 = (n_1/n_2) \sin \theta_i + \nu \lambda_0/(n_2 \Lambda)$. The solid curves show this analytic prediction, while the symbols are maxima from a numerical finite-aperture Fourier evaluation of the same transmitted phase-screen field in Eq. (8); the discrepancy is, again, below approximately 8×10^{-4} . The two signs of ν produce qualitatively different spectral trajectories: the $\nu = +1$ order lies outside the transmitted-medium light cone below 225.1 THz and therefore has no real far-field refraction angle, while the $\nu = -1$ order remains propagating throughout the displayed band and is refracted to the opposite side of the mean surface normal. These results emphasize that the propagating and evanescent cases do not require distinct momentum-matching

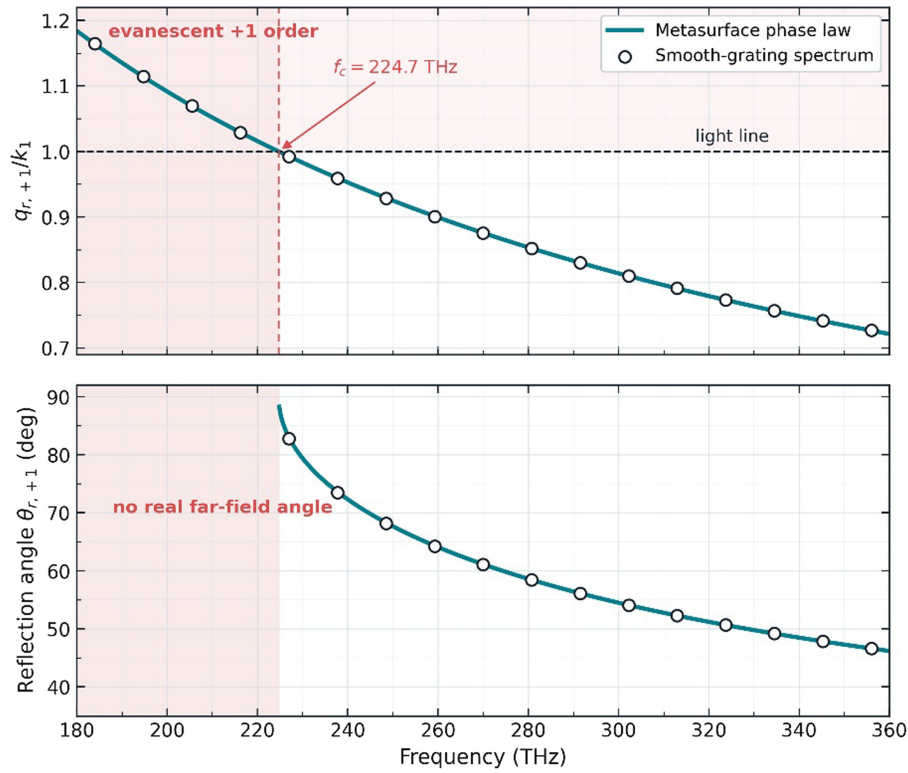


Fig. 2. Numerical check of the reduced phase-screen model for the $\nu = +1$ reflected order. The solid curve is the generalized tangential-momentum prediction, and the symbols are spectral-peak positions from a finite-aperture Fourier evaluation of the same assumed mean-plane field. The order is evanescent below $f_c = 224.7$ THz and propagating above it. Parameters are $n_1 = 1$, $\vartheta_i = 15^\circ$, $\Lambda = 1.80 \mu\text{m}$, and $h_0 = 60 \text{ nm}$.

principles: they are different regimes of the same generalized phase law, distinguished solely by whether $|q_{t,\nu}|$ is smaller or larger than k_2 . Accordingly, the numerical agreement in Figs. 2–4 checks the finite-aperture discretization, analytic Fourier expansion, and peak extraction within the reduced phase-screen model.

The behavior close to the light line is examined more closely in Fig. 4 through the decay constants and penetration depths of the evanescent reflected and transmitted orders. In the evanescent regime, the normal wavenumber is imaginary and may be written as $\kappa_j = i\alpha_j$, with $\alpha_j = \sqrt{q_j^2 - k_j^2}$; consequently, the field penetration depth is $\delta_j = 1/\alpha_j$. Both calculated decay constants decrease continuously to zero as the corresponding cutoff frequency is approached from below, showing that an evanescent order is continuously converted into a propagating one rather than appearing discontinuously. This transition possesses a universal square-root form. Indeed, defining $\epsilon = 1 - f/f_c$, the normalized tangential momentum can be expanded as $q_j/k_j = 1 + A_j\epsilon + O(\epsilon^2)$, where $A_r = 1 - \sin\theta_i$ for reflection and $A_t = 1 - (n_1/n_2)\sin\theta_i$ for refraction. It follows that $\alpha_j/k_0 \simeq n_j\sqrt{2A_j\epsilon}$, and hence $\delta_j/\lambda_0 \simeq [2\pi n_j\sqrt{2A_j}]^{-1}\epsilon^{-1/2}$. The numerical curves in Fig. 4 approach precisely this $-1/2$ power law, demonstrating that the evanescent penetration length diverges at the light-line threshold and providing a quantitative near-field signature of the generalized reflection and refraction laws.

We note that the exponent $-1/2$ follows generically from the square-root normal dispersion $\kappa_j^2 = k_j^2 - q_j^2$

whenever $D_j(f) = q_j(f)^2 - k_j(f)^2$ has a simple zero at the light line: $\alpha_j = \sqrt{D_j} \approx \sqrt{|D'_j(f_c)|}|f - f_c|$, and $\delta_j = 1/\alpha_j \propto |f - f_c|^{-1/2}$. The present correspondence fixes the common cutoff and prefactor through the common $q_j(f)$. A higher-order contact can change the exponent, while absorption, material dispersion, finite aperture, and finite spectral resolution regularize the ideal divergence.

6. DESIGN IMPLICATIONS AND APPLICATIONS

Our correspondence supplies a preliminary inverse design rule. For a target reflected tangential momentum $q_{r,\text{target}}(z)$, Eq. (13) gives $h'(z) = [q_{r,\text{target}}(z) - q_i]/[2k_1\cos\theta_i]$; Eq. (14) gives the analogous transmitted-profile rule when $k_1\cos\theta_i - k_2\cos\theta_t^{(0)}$ is nonzero. For a periodic profile, Λ fixes the momentum increments, angular dispersion, and light-line cutoffs through Eqs. (9) and (10), whereas h_0 controls leading phase-screen order weights. These relations offer starting designs for spectrometers, wavelength demultiplexers, pulse control elements, external-cavity gratings, beam deflectors, and grating couplers. Evanescent-order conditions can also identify ranges for coupling to guided or surface-bound modes and estimate near-field decay lengths; they do not determine the coupling efficiency.

A surface-relief grating is often the direct choice for a fixed, periodic, intrinsically dispersive function over a large aperture, particularly where mature ruling or etching, broad bandwidth, low material loss, or high-power handling are important. A metasurface is generally preferable when the priority is

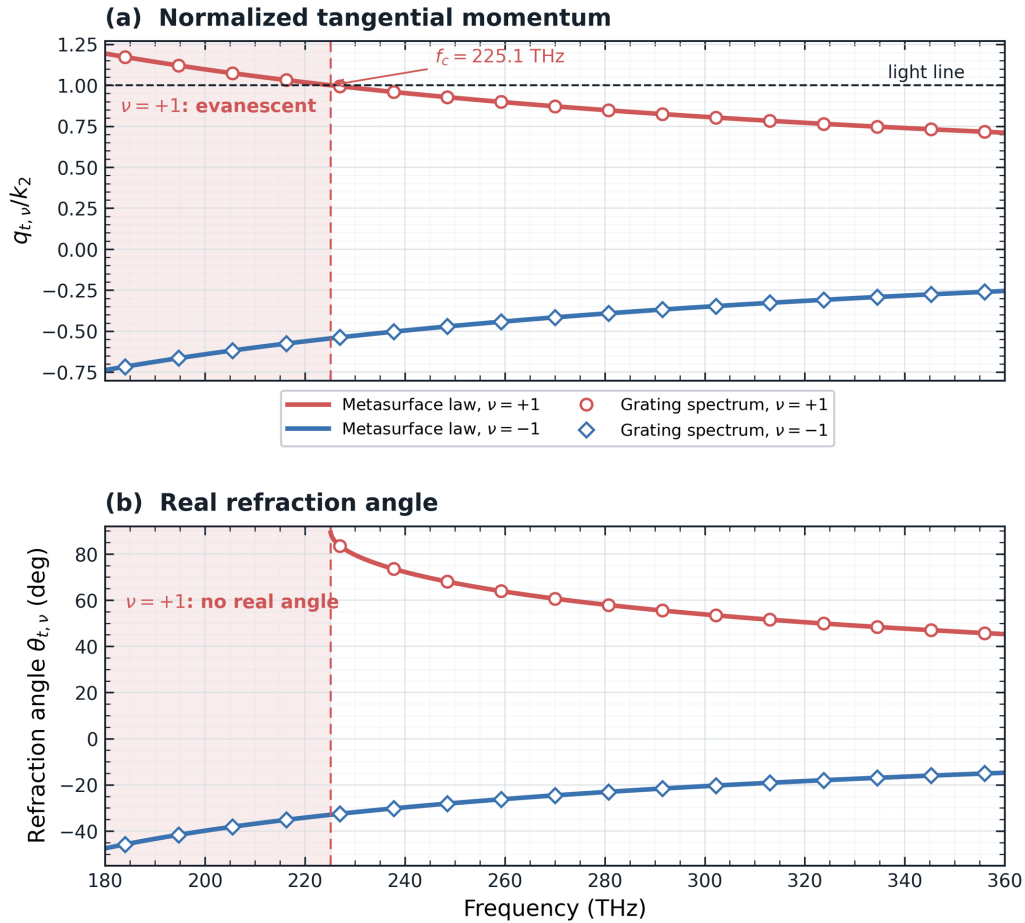


Fig. 3. Numerical corroboration for the $\nu = +1$ and $\nu = -1$ transmitted orders, with $n_1 = 1$, $n_2 = 1.5$, $\theta_i = 20^\circ$, $\Lambda = 1.15 \mu\text{m}$, and $h_0 = 40 \text{ nm}$. (a) Normalized tangential momenta from the common phase-matching law (solid curves) and finite-aperture phase-screen spectra (open symbols). The $+1$ order is evanescent below $f_c = 225.1$ THz. (b) Corresponding real refraction angles; note that no real far-field angle exists for the $+1$ order below cutoff.

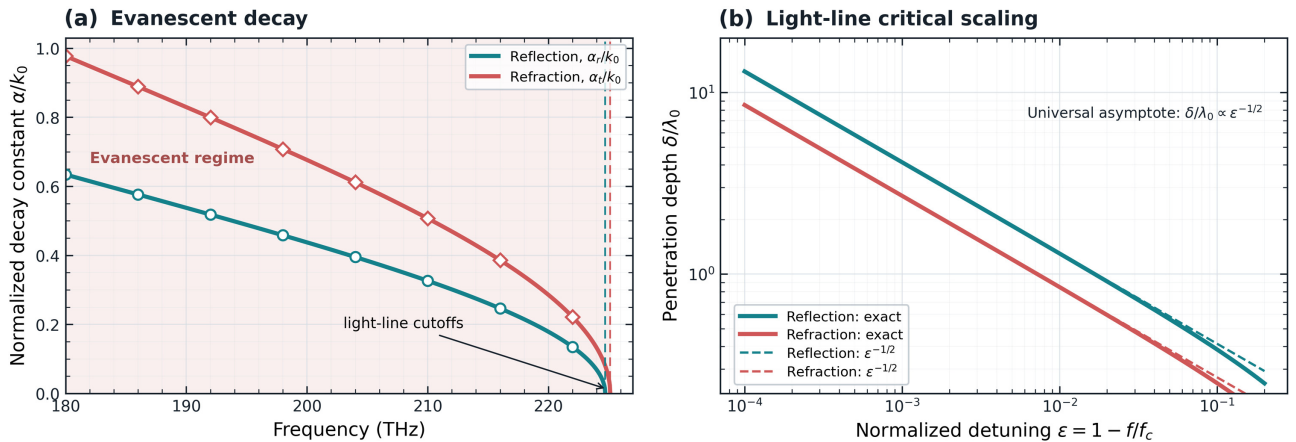


Fig. 4. Evanescent $\nu = +1$ reflected and transmitted orders for the parameter sets of Figs. 2 and 3. (a) The normalized decay constants α_r/k_0 and α_t/k_0 vanish at the respective light-line cutoffs. (b) The exact phase-screen results approach $\delta/\lambda_0 \propto \epsilon^{-1/2}$, where $\epsilon = 1 - f/f_c$.

an ultra-thin planar form, aperiodic or high-numerical-aperture wavefront shaping, holography, simultaneous phase-amplitude-polarization control, or active reconfiguration. We note that these are general tendencies, not exclusive classes; the final choice must compare efficiency, bandwidth, polarization sensitivity, absorption, parasitic

orders, fabrication tolerance, power handling, and system integration.

Based on this analysis, we may now ask what is the conclusion concerning the comparison between phase-gradient metasurfaces and smooth, slowly varying diffraction gratings. The answer is subtle and twofold: at the level of scalar wave kinematics, if what we care about is the angle law for beam steering,

then the generalized metasurface laws [1,5] are, clearly, akin to the *phase laws* governing diffraction gratings/phased arrays. The fundamental physics are not necessarily fundamentally new, i.e., in both cases, the reflection/refraction laws are similar. The strongest uniqueness is at the level of device/platform physics, i.e., that one may now (with metasurfaces) implement a designed phase discontinuity with a physically flat, ultra-thin, subwavelength-structured interface at optical frequencies. This does matter because, as we saw above, in a smooth grating, the phase is tied to the actual height: $\Phi_r(z) = 2k_0 n_1 \cos \theta_i h(z)$, $\Phi_t(z) = k_0(n_1 \cos \theta_i - n_2 \cos \theta_t^{(0)})h(z)$, i.e., we do not have arbitrary independent control of $\Phi(z)$. By contrast, in a metasurface, the phase can be engineered by local scatterers on an almost flat interface, with subwavelength sampling and simultaneous control of phase, amplitude, and polarization. In this sense, gradient metasurfaces can be seen as extending older diffraction-grating surfaces and microwave transmit-array/reflect-array ideas into new spectral regimes with arbitrary wavefront shaping at subwavelength resolution. Overall, the generalized Snell law picture of Ref. [1] is a very good kinematic design rule, but it is not automatically a full Maxwell-consistent solution. This is why later works showed that local phase-gradient metasurface designs can appreciably suffer from parasitic scattering and somewhat limited efficiency, especially for large steering angles, unless nonlocal coupling or bianisotropy are introduced [6–15].

Disclosures. The author declares no conflicts of interest.

Data availability. Data underlying the results presented in this paper are not publicly available at this time but may be obtained from the authors upon reasonable request.

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